MEASURE THEORY AND INTEGRATION 2015/2016 – FINAL EXAM
Instructor: Daniel Valesin

	Q1	Q2	Q3	Q4	Q5	Q6	Free points	$\sum$
Total score:	10	16	16	16	16	16	10	100
Score obtained:							10	

1. Indicate which of the following statements is *necessarily true* (no justification is required in this exercise).

   (a) Let $(\Omega, \mathcal{A})$ be a measurable space and $f : \Omega \to \mathbb{R}$. f is measurable if and only if $|f|$ is measurable.

   (b) If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $f \in \mathcal{L}^\infty(\Omega)$, then $|f| \le \|f\|_\infty$ almost everywhere.

   (c) If A is a Borel measurable subset of $\mathbb{R}$ with empty interior, then the Lebesgue measure of A is zero.

   (d) If A is a Borel measurable subset of $\mathbb{R}$ and the Lebesgue measure of A is zero, then A has empty interior.

   (e) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a measurable function. Then, $f \ge 0$ almost everywhere if and only if $\int_E |f|\, d\mu = \int_E f^+\, d\mu$ for all $E \in \mathcal{A}$.

   (f) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a measurable and nonnegative function. Then, f is integrable if and only if $n \cdot \mu(\{\omega : f(\omega) > n\}) \to 0$ as $n \to \infty$.

   (g) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $p \ge 1$ and $f, g \in \mathcal{L}^p(\Omega)$. Then, $f \cdot g \in \mathcal{L}^p(\Omega)$.

2. Let Ω be a set.

   (a) Show that the collection $\mathcal{A} = \{A \subset \Omega : A \text{ or } A^c \text{ is countable}\}$ is a σ-algebra.

   (b) Show that $\mathcal{A} = \sigma(\{\{x\} : x \in \Omega\})$.

3. (a) Give the definition of outer measures.

   (b) Let $(\Omega, \mathcal{A})$ be a measurable space such that

   $$\text{for all } \omega \in \Omega,\ \{\omega\} \in \mathcal{A}. \qquad (\clubsuit)$$

   A measure μ on $(\Omega, \mathcal{A})$ is called *non-atomic* if $\mu(\{\omega\}) = 0$ for all ω. Prove that $(\mathbb{R}, \mathcal{B})$ satisfies $(\clubsuit)$ and that Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ is non-atomic.

4. (a) Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces, $\mathcal{E} \subset \mathcal{A}_2$ be a collection of sets with $\sigma(\mathcal{E}) = \mathcal{A}_2$ and $f : \Omega_1 \to \Omega_2$ be a function satisfying

   $$f^{-1}(E) \in \mathcal{A}_1 \quad \text{for all } E \in \mathcal{E}.$$

   Show that f is measurable.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that the derivative of f is a measurable function from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{B})$.

5. (a) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ be a nonnegative and measurable function. Show that

$$\nu(A) = \int_A f \, d\mu, \quad A \in \mathcal{A}$$

is a measure.

(b) Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of measurable functions converging pointwise to $f : \Omega \to \mathbb{R}$. Assume that $|f_n| \leq g$ for some $g \in \mathcal{L}^p(\Omega)$ and all n. Show that $f \in \mathcal{L}^p(\Omega)$ and that $\|f_n - f\|_p \to 0$ as $n \to \infty$.

6. Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be measurable spaces. Let μ, μ' be measures on $(\Omega_1, \mathcal{A}_1)$ and ν, ν' be measures on $(\Omega_2, \mathcal{A}_2)$. Assume that

- $\mu(\Omega_1)$, $\mu'(\Omega_1)$, $\nu(\Omega_2)$ and $\nu'(\Omega_2)$ are all finite;
- for all sets $A \in \mathcal{A}_1$ with $\mu(A) = 0$, we have $\mu'(A) = 0$;
- for all sets $B \in \mathcal{A}_2$ with $\nu(B) = 0$, we have $\nu'(B) = 0$.

Prove that for all sets $D \in \mathcal{A}_1 \otimes \mathcal{A}_2$ with $(\mu \otimes \nu)(D) = 0$ we have $(\mu' \otimes \nu')(D) = 0$.